

# Liquidity and market incompleteness

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**Abstract** This note shows that according to Lippman and McCall's (Am Econ Rev 76, 43–55, 1986) operational definition of liquidity, incomplete markets are a necessary condition for illiquidity.

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**JEL Classification** G10 · D83 · G22

## 1 Introduction

Lippman and McCall (1986, henceforth LM) propose an operational measure of liquidity inspired on search theoretical models: an asset's liquidity could be measured by the expectation of the random time  $\tau^*$  at which it is sold under an optimal selling policy. An asset is perfectly liquid if it is optimal to instantaneously transform it into cash:  $E[\tau^*] = 0$ . Cash is hence perfectly liquid by definition, houses are not since they carry an optimal waiting time:  $E[\tau^*] > 0$ .

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The purpose of this note is to point out that under this definition, market incompleteness is a necessary condition for illiquidity: when security markets are complete all assets are perfectly liquid in the LM sense.

The argument is simple, and rests on two well known results for complete market economies, namely (1) the Fundamental Theorem of Asset Pricing,<sup>1</sup> and (2) Fisher and Hirshleifer's Separation Theorem.<sup>2</sup> By (2) the problem of picking a selling strategy is objective in nature, that is, all non-satiated investors –regardless of their preferences for consumption flows across time and states– would pick a policy that maximizes the strategy's current market value. By (1), that value must coincide with the current asset price, for otherwise there would be arbitrage opportunities. Hence, in equilibrium there can be no expected gain from waiting, and selling immediately must be one of the optimal policies. Under this policy all assets verify the property  $E[\tau^*] = 0$ : when security markets are complete all assets are perfectly liquid.

## 2 The model

Let  $(\Omega, \sigma, p, \{H_t\})$  be a filtered probability space:  $\Omega$  is a set of states,  $\{H_t\}$  a filtration,<sup>3</sup>  $\sigma$  is the  $\sigma$ -algebra constructed by the union of all algebras generated by each  $H_t$ , and  $p$  a probability measure over  $\sigma$ . Time is discrete:  $t = 0, 1, 2, \dots$ . The cell of partition  $H_t$  containing state  $\omega$  is denoted by  $h_t(\omega)$ . A stochastic process  $\{X_t\}$  is said to be adapted to  $\{H_t\}$  (or adapted for short) if  $\omega' \in h_t(\omega) \Rightarrow X_t(\omega) = X_t(\omega')$  for all  $t$ .

Assets are indexed by  $k \in K$ ;  $\{Q_t^k\}$  is the adapted stochastic process representing asset  $k$ 's (net) price, and  $\{R_t^k\}$  its payoff or dividend. By the Fundamental Theorem of Asset Pricing, asset market completeness implies the existence of an adapted stochastic process  $\{\xi_t\}$  such that for all  $k$ ,

$$Q_0^k = \sum_{t \geq 0} \sum_{\omega \in \Omega} \xi_t(\omega) R_t^k(\omega).$$

The stochastic process  $\{\xi_t\}$  is known as the stochastic discount factor.

In the search-theoretical-based liquidity model of LM the focus is on optimal strategies for selling an asset  $k$ ,  $S = \{S_t^k\}$ , where  $S_t^k$  is an indicator function taking the value 1 at  $h_t(\omega)$  if at that event the asset is sold, and 0 otherwise. Strategy  $S$  is feasible if it is adapted to  $\{H_t\}$  and if  $\forall \omega \in \Omega$ :

$$\sum_{t \geq 0} S_t^k(\omega) = 1.$$

Let  $\mathcal{S}$  denote the set of feasible strategies.

<sup>1</sup> See for instance Barucci (2003), page 164.

<sup>2</sup> See for instance Eichberger and Harper (1997), page 145.

<sup>3</sup> A filtration is a collection of partitions of  $\Omega$  where each  $H_t$  is a refinement of  $H_{t-1}$ .

A selling strategy  $S$  is associated to the stopping time  $\tau_S$  defined by:

$$\tau_S(\omega) = \sum_{t \geq 0} S_t^k(\omega) t$$

that specifies for each state  $\omega$  the date at which the asset is sold according to  $S$ .

LM consider the selling strategy that maximizes the expected value of:

$$\beta^\tau Q_\tau^k(\omega) + \sum_{i=1}^{\tau} \beta^i R_i^k(\omega)$$

which incidentally coincides with the maximization of the discounted, expected life-time utility function:

$$U(\{C_t(\omega)\}) = \sum_{t \geq 0} \sum_{\omega \in \Omega} \beta^t u(C_t(\omega))$$

if the individual consumes whatever he gets in the form of dividend or revenue from the sale of the asset,  $C_t(\omega) = Q_t(\omega)S_t(\omega) + (1 - S_t(\omega))R_t(\omega)$ , and the instantaneous utility is linear:  $u(C) = C$ .

Let  $U_r(S)$  denote the utility of the consumption flow generated by strategy  $S$  under market regime  $r$ , that is, under complete markets:

$$\begin{aligned} U_C(S) = & \max_{\{C_t(\omega)\}} U(\{C_t(\omega)\}) \\ & \text{subject to} \\ & \sum_{t \geq 0} \sum_{\omega \in \Omega} \xi_t(\omega) C_t(\omega) \leq W(S) \end{aligned}$$

(where

$$W(S) \equiv \sum_{\omega \in \Omega} \sum_{t=0}^{\tau_S(\omega)} \xi_t(\omega) \left[ (1 - S_t^k(\omega)) R_t^k(\omega) + S_t^k(\omega) Q_t^k(\omega) \right]$$

is the date-0 value of the flows generated by the strategy  $S$ ), and when only asset  $k$  is available (i.e., the case considered by LM):

$$U_I(S) = \sum_{\omega \in \Omega} \sum_{t=0}^{\tau_S(\omega)} \beta^t u(Q_t(\omega)S_t(\omega) + (1 - S_t(\omega))R_t(\omega))$$

Under special assumptions that guarantee the existence of a unique optimal policy  $S^* \in \mathcal{S}$  with the reservation-price property (that is, sell the first time  $Q_t(\omega)$  is higher than a fixed threshold level  $\bar{Q}$ ), LM obtain a solution  $\tau^*$  that depends, among other variables, on the investor's discount factor  $\beta$ .

We embed their search environment into a Walrasian framework: at each time  $t$  the offer received is the going Walrasian market price for asset  $k$ ,  $Q_t^k(\omega)$ . Hence, unlike LM we do not consider the possibility of random arrival of offers, nor the possibility of taking offers that were received in the past (the case of recall). Moreover, we drop their assumption of a negative, constant dividend flow ( $R_t(\omega) = -c$ ) where  $c$  receives the interpretation of a holding cost; the fact that the dividend they consider is negative is immaterial in view of the linearity of the utility function, but when embedded in our model it becomes at odds with the price being both, positive and stochastic. The main difference however lies elsewhere: they consider the market for asset  $k$  in isolation, while we consider the case where at all times and events there is a complete security market.

### 3 The result

The first thing to note is that under market completeness Fisher and Hirshleifer's Separation Theorem holds, whereby the optimal selling strategy is independent of the investor's preferences—including his degree of impatience, represented by his discount factor. In effect, Fisher and Hirshleifer's theorem in our case reads:

**Lemma**  $\arg \max_{S \in \mathcal{S}} U_C(S) = \arg \max_{S \in \mathcal{S}} W(S)$  for every increasing  $u(C)$ .

*Proof* Consider the optimal consumption plan  $\{C_t^*(\omega)\}$  under a wealth constraint of  $W^*$ . Because the plan is optimal, and because of non satiation, any other plan that is preferred to it must be more expensive:  $U(\{C_t^{**}(\omega)\}) > U(\{C_t^*(\omega)\}) \Rightarrow W^{**} > W^*$ . Hence,  $U$  is increasing in  $W$ . It follows that the optimal strategy  $S$  maximizes  $W(S)$ .  $\square$

The second part of the argument asserts that the strategy  $S^0$ , selling immediately:

$$\forall \omega \in \Omega, \quad S_t^0(\omega) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{if } t > 0 \end{cases}$$

associated to the stopping time:

$$\tau(\omega) = 0 \quad \forall \omega \in \Omega,$$

is in the set  $\arg \max_{S \in \mathcal{S}} W(S)$  :

**Theorem**  $S^0 \in \arg \max_{S \in \mathcal{S}} W(S)$

*Proof* Observe that the value of strategy  $S^0$  is  $W(S^0) = Q_0^k(\omega)$ , the date-0 asset  $k$  price. Suppose, on the contrary, that there exists a feasible selling strategy  $S' \in \mathcal{S}$  such that  $W(S') > Q_0^k(\omega)$ . Then the portfolio made of  $-[(1 - S'_t(\omega))R_t^k(\omega) + S'_t(\omega)Q_t^k(\omega)]$  units of the Arrow security that pays 1 unit of account if event  $h_t(\omega)$  materializes and nothing otherwise, and +1 unit of asset  $k$  is an arbitrage opportunity. In effect, buying the asset at time 0 and executing policy  $S'$  is feasible, it ensures breaking even at the close of the position at all states, and costs at date 0 ( $Q_0^k(\omega) - W(S') < 0$ ): it is an arbitrage opportunity.  $\square$

Put another way: the Separation theorem asserts that if one investor expects a gain from waiting, then all investors should expect the same gain, and therefore asset prices were not in equilibrium in the first place.

#### 4 Further remarks

The result is stated in terms of date-0 decisions; it generalizes easily to every date- $t$  decision because of the well known time consistency properties of expected utility, but the notation required is more cumbersome.

The argument is actually stronger than stated: under complete markets the no arbitrage condition implies that any selling strategy must have the same value, the current market price:  $S = \arg \max_{S \in \mathcal{S}} W(S)$ . Hence, there is an optimal selling strategy for every value of  $E[\tau]$  we may wish for. In cases like this, when there are many optimal policies, it would seem that the spirit of LM's definition requires us to pick the lowest expected random time policy.

The conclusion to draw from this result is that the proper study of liquidity belongs to the realm of incomplete market models.

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